B.Sc Part I

Date - 24-04

(1)

Topic - Resolution of Trigonometrical Functions in to Factors (Trigonometry)

Contd.—

Resolution of $\cos\theta$ into factors (Infinite Product)

Since we have $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\Rightarrow \cos\theta = \frac{\sin 2\theta}{2\sin\theta}$$

As we have $\sin\theta = \theta \prod_{r=1}^{\infty} 1 - \frac{\theta^2}{r^2\pi^2}$

$$\therefore \cos\theta = \frac{2\theta \prod_{r=1}^{\infty} 1 - \frac{2^2\theta^2}{r^2\pi^2}}{2\cdot\theta \prod_{r=1}^{\infty} 1 - \frac{\theta^2}{r^2\pi^2}}$$

$$= \frac{\left[1 - \frac{4\theta^2}{1^2\pi^2}\right]\left[1 - \frac{4\theta^2}{2^2\pi^2}\right]\left[1 - \frac{4\theta^2}{3^2\pi^2}\right]\left[1 - \frac{4\theta^2}{5^2\pi^2}\right]\cdots\infty}{\left[1 - \frac{\theta^2}{1^2\pi^2}\right]\left[1 - \frac{\theta^2}{2^2\pi^2}\right]\left[1 - \frac{\theta^2}{3^2\pi^2}\right]\left[1 - \frac{\theta^2}{4^2\pi^2}\right]\cdots\infty}$$

Clearly we see here
2nd factor of N^r get cancel with first factor of D^r
4th factor of N^r cancel with 2nd factor of D^r
and so on and finally we have

$$\cos\theta = \left[1 - \frac{4\theta^2}{\pi^2}\right]\left[1 - \frac{4\theta^2}{3^2\pi^2}\right]\left[1 - \frac{4\theta^2}{5^2\pi^2}\right]\cdots$$

$$= \prod_{r=1}^{\infty} 1 - \frac{4\theta^2}{(2r-1)^2\pi^2}$$

2. Resolution of $\sinh\theta$ as infinite product.

Since we know

$$\sin\theta = \theta\left[1 - \frac{\theta^2}{\pi^2}\right]\left[1 - \frac{\theta^2}{2^2\pi^2}\right]\left[1 - \frac{\theta^2}{3^2\pi^2}\right]\cdots\infty \quad (1)$$

Since $\sinh\theta = -i\sin i\theta$

$$= -i(i\theta)\left[1 - \frac{i^2\theta^2}{\pi^2}\right]\left[1 - \frac{i^2\theta^2}{2^2\pi^2}\right]\left[1 - \frac{i^2\theta^2}{3^2\pi^2}\right]\cdots\infty$$

using $i^2 = -1$

$$= \theta\left[1 + \frac{\theta^2}{\pi^2}\right]\left[1 + \frac{\theta^2}{2^2\pi^2}\right]\left[1 + \frac{\theta^2}{3^2\pi^2}\right]\cdots\infty$$

$$= \theta \prod_{r=1}^{\infty}\left(1 + \frac{\theta^2}{r^2\pi^2}\right)$$

3. Resolution of $\cosh\theta$ as infinite product.

Since we know

$$\cos\theta = \left[1-\frac{4\theta^2}{\pi^2}\right]\left[1-\frac{4\theta^2}{3^2\pi^2}\right]\left[1-\frac{4\theta^2}{5^2\pi^2}\right] \cdots\cdots \infty$$

As $\cosh\theta = \cos i\theta$

$$\cos i\theta = \left[1-\frac{4i^2\theta^2}{\pi^2}\right]\left[1-\frac{4i^2\theta^2}{3^2\pi^2}\right]\left[1-\frac{4\theta^2 i^2}{5^2\pi^2}\right] \cdots \infty$$

using $i^2=-1$ and $\cos i\theta = \cosh\theta$.

we have $\cosh\theta = \left[1+\frac{4\theta^2}{\pi^2}\right]\left[1+\frac{4\theta^2}{3^2\pi^2}\right]\left[1+\frac{4\theta^2}{5^2\pi^2}\right] \cdots$

$$\cosh\theta = \prod_{r=1}^{\infty}\left[1+\frac{4\theta^2}{(2r-1)^2\pi^2}\right]$$

4. Show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$

(iii) $\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \cdots \infty = \frac{6\times(2\pi)^6}{\lfloor 9}$

(i) Since we have

$$\sin\theta = \theta\left[1-\frac{\theta^2}{\pi^2}\right]\left[1-\frac{\theta^2}{2^2\pi^2}\right]\left[1-\frac{\theta^2}{3^2\pi^2}\right]\cdots\cdots \infty$$

$$\Rightarrow \frac{\sin\theta}{\theta} = \left[1-\frac{\theta^2}{\pi^2}\right]\left[1-\frac{\theta^2}{2^2\pi^2}\right]\left[1-\frac{\theta^2}{3^2\pi^2}\right]\cdots \quad \text{——— (i)}$$

$$\left[\text{since } \sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \frac{\theta^7}{\lfloor 7}\cdots\right.$$

$$\left. = \theta\left[1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\frac{\theta^6}{\lfloor 7}\cdots\right]\right]$$

$$\Rightarrow \frac{\sin\theta}{\theta} = 1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\frac{\theta^6}{\lfloor 7}+\cdots \quad \text{——— (ii)}$$

From (I) and (II)

$$\left[1-\frac{\theta^2}{\pi^2}\right]\left[1-\frac{\theta^2}{2^2\pi^2}\right]\left[1-\frac{\theta^2}{3^2\pi^2}\right]\cdots = 1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\frac{\theta^6}{\lfloor 7}+\cdots$$

$$\Rightarrow \left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots = 1-\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\frac{\theta^6}{\lfloor 7}\cdots\right]$$

Taking log of both sides.

$$\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\log\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\cdots$$
$$=\log\left[1-\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\frac{\theta^6}{\lfloor 7}\cdots\right)\right]$$

Using $\log(1-x) = -\left[x+\frac{x^2}{2}+\frac{x^3}{3}\cdots\right]$

$$-\left\{\left[\frac{\theta^2}{\pi^2}+\frac{1}{2}\frac{\theta^4}{\pi^4}+\frac{1}{3}\frac{\theta^6}{\pi^6}\cdots\right]+\left(\frac{\theta^2}{2^2\pi^2}+\frac{1}{2}\cdot\frac{\theta^4}{2^4\pi^4}+\frac{1}{3}\frac{\theta^6}{2^6\pi^6}\cdots\right)\right.$$
$$\left.+\left(\frac{\theta^2}{3^2\pi^2}+\frac{1}{2}\frac{\theta^4}{3^4\pi^4}+\frac{1}{3}\cdot\frac{\theta^6}{3^6\pi^6}\cdots\right)\right\}$$
$$=-\left[\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\frac{\theta^6}{\lfloor 7}\cdots\right)+\frac{1}{2}\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\frac{\theta^6}{\lfloor 7}\cdots\right)^2\right.$$
$$\left.+\frac{1}{3}\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\frac{\theta^6}{\lfloor 7}\cdots\right)^3+\cdots\right]$$

Equating Co-efficient of θ^2 on both sides

$$\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\cdots = \frac{1}{\lfloor 3}$$

$$\therefore 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{\lfloor 3} = \frac{\pi^2}{6}$$

Equating co-efficient of θ^4

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots = \frac{\pi^4}{90}$$

and Equating the co-efficient of θ^6

$$1/1^6+\frac{1}{2^6}+\frac{1}{3^6}+\cdots = \frac{6\cdot(2\pi)^6}{\lfloor 9}$$

(4)

Show that

(i) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots = \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$

We have $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$

$= \left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}\cdots\right]-\left[\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots\right]$

$= \left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\cdots\infty\right]-\frac{1}{2^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$

$= \frac{\pi^2}{6}-\frac{1}{4}\frac{\pi^2}{6} = \frac{\pi^2}{6}\left[1-\frac{1}{4}\right] = \frac{\pi^2}{\cancel{6}2}\times\frac{\cancel{3}}{4} = \frac{\pi^2}{8}$

(ii) Show that

$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots = \frac{\pi^4}{96}$

$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$

$= \left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}\cdots\right]-\left[\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\cdots\right]$

$= \left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\right]-\frac{1}{2^4}\left[1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\right]$

$= \frac{\pi^4}{90}-\frac{1}{16}\frac{\pi^4}{90}$

$= \frac{\pi^4}{90}\left[1-\frac{1}{16}\right]$

$= \frac{\pi^4}{\cancel{90}6}\times\frac{\cancel{15}}{16} = \frac{\pi^4}{96}$